TOPIC PLAN		
Partner organization	UNS	
Topic	Function of Several Variables	
Lesson title	Partial Derivatives	
Learning objectives	✓ Students will be able to determine partial derivatives of functions of several variables; ✓ Students will acquire and deal with derivatives of a function; ✓ Students will be able to deal with different problems in everyday life, which require finding partial derivatives of a given function; ✓ Students are encouraged to use technology and different software in their work, while considering problem-based situations.	**Strategies/Activities** ☐Graphic Organizer ☐ Think/Pair/Share ☐Modeling ☐Collaborative learning ☐Discussion questions ☐Project based learning ☐Problem based learning **Assessment for learning** ☐Observations ☐Conversations ☐Work sample ☐Conference ☐Check list ☐Diagnostics
Aim of the lecture / Description of the practical problem	The aim of the lecture is to make students able to calculate partial derivatives of a function and apply the derivatives to calculate approximation of functions. The teacher gives the next problem to the students: 1. Find the equation of the tangent plane to $z = x^2\cos(\pi y) - \frac{6}{xy^2}$ at $(2, -1)$. 2. Find the linear approximation to $z = 4x^2 - ye^{2x+y}$ at $(-2, 4)$.	
Previous knowledge assumed:	- functions - algebraic equations - differentiating techniques	

Introduction / Theoretical basics	Consider the function f given by $$z = f(x,y) = x^2y^3 + xy + 4y^2$$ Suppose for the moment that we fix y at 3. Then $$f(x,3) = x^2 3^3 + x3 + 4\,3^2 = 27x^2 + 3x + 36$$ Note that we now have a function of only one variable. Taking the first derivative with respect to x, we have $$54x + 3$$ In general, without replacing y with a specific number, we can consider y fixed. Then f becomes a function of x alone, and we can calculate its derivative with respect to x. This derivative is called the partial derivative of f with respect to x. Notation for this partial derivative is $$\frac{\partial f}{\partial x} \text{ or } d_x$$ Now, let's again consider the function $$z = f(x,y) = x^2y^3 + xy + 4y^2$$ The color blue indicates the variable x when we fix y and treat it as a constant. The expressions y, and are then also treated as constants. We have $$\frac{\partial f}{\partial x} = 2xy^3 + y$$ Similarly, we find $\frac{\partial f}{\partial y}$ by fixing x (treating it as a constant) and calculating the derivative with respect to y. From $$z = f(x,y) = x^2y^3 + xy + 4y^2$$ We get $$\frac{\partial f}{\partial y} = 3x^2y^2 + x + 8y$$ A definition of partial derivatives is as follows **DEFINITION** For $z = f(x,y)$, the partial derivatives with respect to x and y are $$\frac{\partial z}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$ and $$\frac{\partial z}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h}$$ We can find partial derivatives of functions of any number of variables. Since we can apply the theorems for finding derivatives presented earlier, we will rarely need to use the definition to find a partial derivative. **EXAMPLE 1** For $w = x^2 - xy + y^2 + 2yz + z$ find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$. *Solution* In order to find $\frac{\partial w}{\partial x}$ we regard x as the variable and treat y and z as constants. From $$w = x^2 - xy + y^2 + 2yz + z$$	**Assessment as learning** ☐Self-assessment ☐Peer-assessment ☐Presentation ☐Graphic Organizer ☐Homework **Assessment of learning** ☐Test ☐Quiz ☐Presentation ☐Project ☐Published work

we get

$$\frac{\partial w}{\partial x} = 2x - y$$

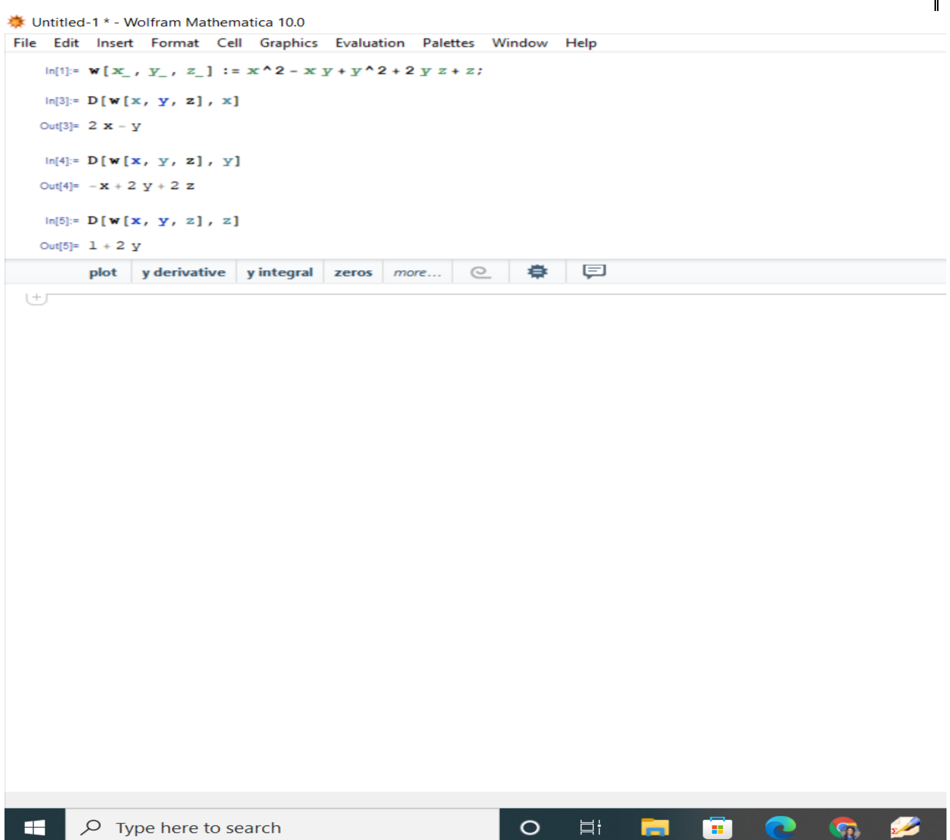
To find $\frac{\partial w}{\partial y}$ we regard y as the variable and treat x and z as constants. We get

$$\frac{\partial w}{\partial y} = -x + 2y + 2z$$

To find we regard z as the variable and treat x and y as constants. We get

$$\frac{\partial w}{\partial z} = 2y + 1$$

Students can calculate these derivatives using Mathematica on the following way



Quick Check 1

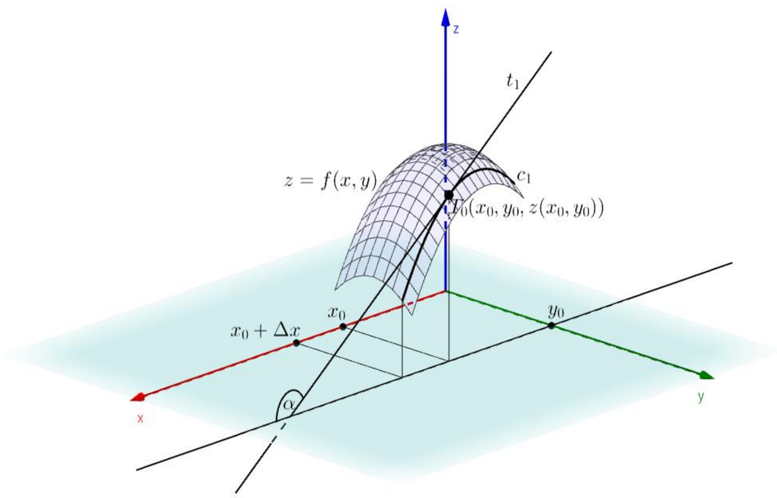
***For** $u = x^2y^3z^4$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.*

We will often make use of a simpler notation: f_x for the partial derivative of f with respect to x and f_y for the partial derivative of f with respect to y. Similarly, if $z = f(x, y)$, then z_x represents the partial derivative of z with respect to x, and z_y represents the partial derivative of z with respect to y.

The Geometric Interpretation of Partial Derivatives

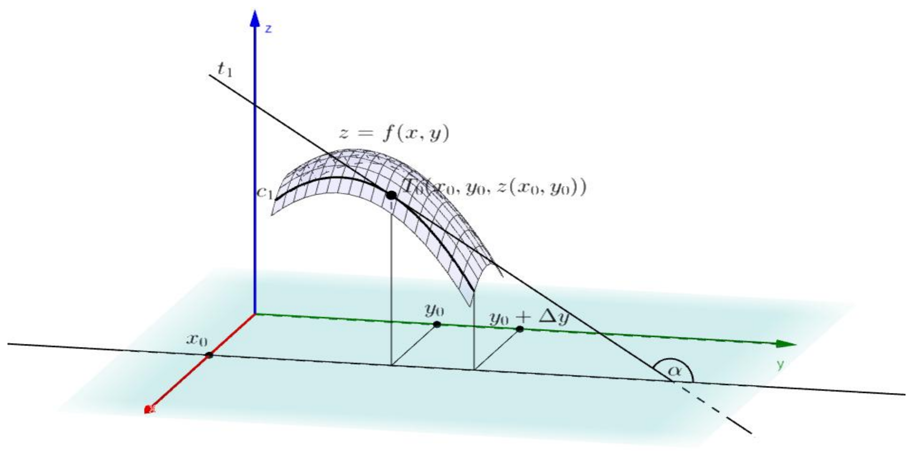
The graph of a function of two variables $z = f(x, y)$ is a surface S, which might have a graph similar to the one shown to the right, where each input pair (x, y) in the domain D has only one output, $z = f(x, y)$.

Slope is $\frac{\partial z}{\partial x}$ (picture below)



Now suppose that we hold y fixed at the value y_0. The set of all points for which $y = y_0$ is a plane parallel to the xz-plane; thus, when y is fixed at y_0, x and z vary along that plane, as shown to the right. The plane in the figure cuts the surface along the curve C_1. The partial derivative f_x gives the slope of tangent lines to this curve, in the positive x-direction. Similarly, if we hold x fixed at the value x_0, we obtain a curve C_2 as shown to the right. The partial derivative f_y gives the slope of tangent lines to this curve, in the positive y-direction.

Slope is $\frac{\partial z}{\partial y}$ (picture below)



An Economics Application: The Cobb–Douglas Production Function

One model of production that is frequently considered in business and economics is the Cobb–Douglas production function:

$$p(x, y) = Ax^a y^{1-a}, \text{ for } A > 0 \text{ and } 0 < a < 1,$$

where p is the number of units produced with x units of labor and y units of capital. (Capital is the cost of machinery, buildings, tools, and other supplies.) The partial derivatives

$$\frac{\partial p}{\partial x} \text{ and } \frac{\partial p}{\partial y}$$

are called, respectively, the *marginal productivity of labor* and the *marginal productivity of capital*.

EXAMPLE 2 A cellular phone company has the following production function for a smart phone:

$$p(x, y) = 50x^{2/3}y^{1/3}$$

where p is the number of units produced with x units of labor and y units of capital.

a) Find the number of units produced with 125 units of labor and 64 units of capital.

b) Find the marginal productivities.

c) Evaluate the marginal productivities at and $x = 125$ and $y = 64$.

Solution

a) $p(125{,}64) = 50(125)^{\frac{2}{3}}(64)^{1/3} = 5000$ units

b) Marginal productivity of labor is $\frac{\partial p}{\partial x} = p_x = 50\frac{2}{3}x^{-1/3}y^{1/3} = \frac{100y^{1/3}}{3x^{1/3}}$,

Marginal productivity of capital is
$\frac{\partial p}{\partial y} = p_y = 50\frac{1}{3}x^{2/3}y^{-2/3} = \frac{50x^{2/3}}{3y^{2/3}}$

c) For 125 units of labor and 64 units of capital, we have

$$p_x(125{,}64) = 26\frac{2}{3} \text{ and } p_y(125{,}64) = 26\frac{1}{24}$$

A Cobb–Douglas production function is consistent with the law of diminishing returns. That is, if one input (either labor or capital) is held fixed while the other increases infinitely, then production will eventually increase at a decreasing rate. With such functions, it also turns out that if a certain maximum production is possible, then the expense of more labor, for example, may be required for that maximum output to be attainable.

TANGENT PLANES AND LINEAR APPROXIMATIONS

Earlier we saw how the two partial derivatives f_x and f_y can be thought of as the slopes of traces. We want to extend this idea out a little in this section. The graph of a function $z = f(x, y)$ is a surface in three dimensional space and so we can now start thinking of the plane that is "tangent" to the surface as a point.

Let's start out with a point (x_0, y_0) and let's let C_1 represent the trace to $f(x, y)$ for the plane $y = y_0$ (*i.e.* allowing x to vary with y held fixed) and we'll let C_2 represent the trace to $f(x, y)$ for the plane $x = x_0$ (*i.e.* allowing y to vary with x held fixed). Now, we know that $f_x(x_0, y_0)$ is the slope of the tangent line to the trace C_1 and $f_y(x_0, y_0)$is the slope of the tangent line to the trace C_2. So, let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

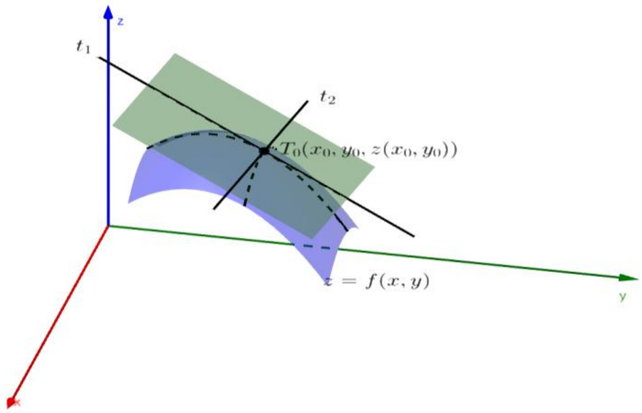
The tangent plane will then be the plane that contains the two lines L_1 and L_2.

Geometrically this plane will serve the same purpose that a tangent line did in Calculs I. A tangent line to a curve was a line that just touched the curve at that point and was "parallel" to the curve at the point in question. Well tangent planes to a surface are planes that just touch the surface at the point and are "parallel" to the surface at the point. Note that this gives us a point that is on the plane. Since the tangent plane and the surface touch at (x_0, y_0) the following point will be on both the surface and the plane.

$$(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$$

What we need to do now is determine the equation of the tangent plane.

Tangent plane



We know that the general equation of a plane is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point that is on the plane, which we have. Let's rewrite this a little. We'll move the x terms and y terms to the other side and divide both sides by c. Doing this gives,

$$z - z_0 = -\frac{a}{c}(x - x_0) - \frac{b}{c}(y - y_0)$$

Now, let's rename the constants to simplify up the notation a little. Let's rename them as follows,

$$A = -\frac{a}{c} \qquad B = -\frac{b}{c}$$

With this renaming the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

and we need to determine values for A and B.

Let's first think about what happens if we hold y fixed, *i.e.* if we assume that $y = y_0$. In this case the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0)$$

This is the equation of a line and this line must be tangent to the surface at (x_0, y_0) (since it's part of the tangent plane). In addition, this line assumes that $y = y_0$ (*i.e.* fixed) and A is the slope of this line. But if we think about it this is exactly what the tangent to C_1 is, a line tangent to the surface at (x_0, y_0) assuming that $y = y_0$. In other words,

$$z - z_0 = A(x - x_0)$$

is the equation for L_1 and we know that the slope of L_1 is given by $f_x(x_0, y_0)$. Therefore, we have the following,

$$A = f_x(x_0, y_0)$$

If we hold x fixed at $x = x_0$ the equation of the tangent plane becomes,

$$z - z_0 = B(y - y_0)$$

However, by a similar argument to the one above we can see that this is nothing more than the equation for L_2 and that it's slope is B or $f_y(x_0, y_0)$. So,

$$B = f_y(x_0, y_0)$$

The equation of the tangent plane to the surface given by $z = f(x, y)$ at $(x_0, y_0)$is then,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Also, if we use the fact that $z_0 = f(x_0, y_0)$ we can rewrite the equation of the tangent plane as,

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We will see an easier derivation of this formula (actually a more general formula) in the next section so if you didn't quite follow this argument hold off until then to see a better derivation.

EXAMPLE 3 Find the equation of the tangent plane to $z = \ln(2x + y)$ at $(-1,3)$.

Solution There really isn't too much to do here other than taking a couple of derivatives and doing some quick evaluations.

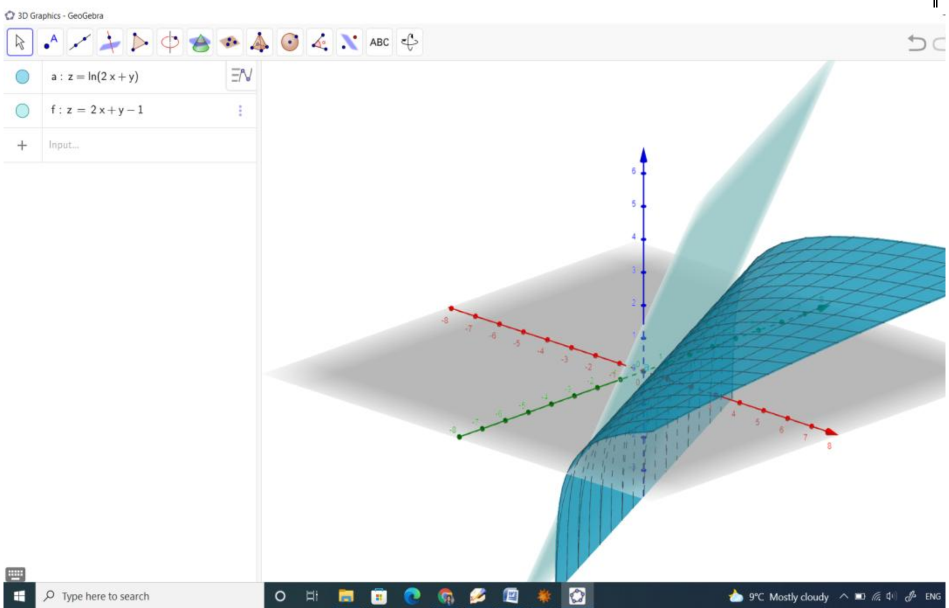
$$f_x(x, y) = \frac{2}{2x+y} \qquad f_x(-1,3) = 2$$

$$f_y(x, y) = \frac{1}{2x+y} \qquad f_y(-1,3) = 1$$

The equation of the plane is then,

$$z - 0 = 2(x + 1) + 1(y - 3)$$

$$z = 2x + y - 1$$



One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. Because of this we define the **linear approximation** to be,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and as long as we are “near” (x_0, y_0) then we should have that,

$$f(x, y) \approx \mathrm{L(x, y)}$$

EXAMPLE 4 Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

Solution So, we’re really asking for the tangent plane so let’s find that

$$f_x(x, y) = \frac{x}{8} \qquad f_x(-4,3) = -\frac{1}{2}$$
$$f_y(x, y) = \frac{2y}{9} \qquad f_y(-4,3) = \frac{2}{3}$$

The tangent plane, or linear approximation, is then,

$$L(x, y) = 5 - \frac{1}{2}(x + 4) + \frac{2}{3}(y - 3)$$

Action		
Materials / equipment / digital tools / software	*<u>The materials for learning</u>* are given as a part of references of the end from this topic plan; *<u>Equipment</u>*: classroom, whiteboard, marker in different colours; *<u>Digital tools</u>*: laptop, projector; *<u>Software</u>*: Geogebra, Mathematica.	
Consolidation	With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn what is a partial derivative of a function and how to calculate it. They can learn how to apply differentiation and derivatives to approximate functions. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	

Reflections and next steps

Activities that worked	**Parts to be revisited**
Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.

References

[1] J. Stewart, Calculus, Thomson Learning, China, 2006.
[2] M. L. Bittinger, D. J. Ellenbogen and S.A. Surgent, “Calculus and its applications”, Addison-Wesley, 2012.
[3] T. Došenović, A. Takači, D. Rakić, Udžbenik iz Matematike II za studente Tehnološkog fakulteta, Univerzitet u Novom Sadu, 2017.
[4] https://tutorial.math.lamar.edu/classes/calciii/TangentPlanes.aspx